

On stability of exponential cosmological solutions with non-static volume factor in the Einstein-Gauss-Bonnet model

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Abstract

A $(n + 1)$ -dimensional gravitational model with Gauss-Bonnet term and cosmological constant term is considered. When ansatz with diagonal cosmological metrics is adopted, the solutions with exponential dependence of scale factors: $a_i \sim \exp(v^i t)$, $i = 1, \dots, n$, are analysed for $n > 3$. We study the stability of the solutions with non-static volume factor, i.e. if $K(v) = \sum_{k=1}^n v^k \neq 0$. We prove that under certain restriction R imposed solutions with $K(v) > 0$ are stable while solutions with $K(v) < 0$ are unstable. Certain examples of stable solutions are presented. We show that the solutions with $v^1 = v^2 = v^3 = H > 0$ and zero variation of the effective gravitational constant are stable if the restriction R is obeyed.

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1 Introduction

This paper is devoted to D -dimensional gravitational model with the so-called Gauss-Bonnet term. It is governed by the action

$$S = \int_M d^D z \sqrt{|g|} \{ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \}, \quad (1.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M , $\dim M = D$, $|g| = |\det(g_{MN})|$ and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4R_{MN} R^{MN} + R^2 \quad (1.2)$$

is the quadratic ‘‘Gauss-Bonnet term’’ and Λ is cosmological term. Here α_1 and α_2 are non-zero constants. The appearance of the Gauss-Bonnet term was motivated by string theory [1, 2, 3].

At present, the so-called Einstein-Gauss-Bonnet (EGB) gravitational model which is governed by the action (1.1) and its modifications are intensively used in cosmology, see [4] - [23] and references therein, e.g. for explanation of accelerating expansion of the Universe following from supernovae (type Ia) observational data [24, 25, 26].

Here we consider the cosmological solutions with diagonal metrics governed by n scale factors depending upon one variable, where $n > 3$; $D = n + 1$. We study the stability of solutions with exponential dependence of scale factors with respect to the synchronous time variable t

$$a_i(t) \sim \exp(v^i t), \quad (1.3)$$

$i = 1, \dots, n$. In our analysis we restrict ourselves by a class of perturbations which depend on t and do not disturb the diagonal form of the metric.

For possible physical applications solutions describing an exponential isotropic expansion of 3-dimensional flat factor-space, i.e. with

$$v^1 = v^2 = v^3 = H > 0, \quad (1.4)$$

and small enough variation of the effective gravitational constant G are of interest. We remind that G (for $4d$ metric in Jordan frame, see Remark 4 in Section 4) is proportional to the inverse volume scale factor of the internal space, see [27, 28, 29] and refs. therein. Due to experimental data, the variation of G is allowed at the level of 10^{-13} per year and less. The most stringent limitation on \dot{G} (coming from the set of ephemerides) was obtained in ref. [30]

$$\dot{G}/G = (0.16 \pm 0.6) \cdot 10^{-13} \text{ year}^{-1} \quad (1.5)$$

allowed at 95% confidence (2σ).

Here we reduce the set of cosmological equations to the (mixed) set of algebraic and differential equations

$$f_0(h) = 0, \quad (1.6)$$

$$f_i(\dot{h}, h) = 0. \quad (1.7)$$

where $h = h(t) = (h^i(t) = \dot{a}_i(t)/a_i(t))$ is the set of so-called “Hubble-like” parameters corresponding to scale factors $a_i(t)$; $f_0(h)$ and $f_i(\dot{h}, h)$ are polynomials of the fourth order in h^i ; $f_i(\dot{h}, h)$ are polynomials of the first order in \dot{h}^i . The fixed point solutions $h^i(t) = v^i$ ($i = 1, \dots, n$) correspond to exponential solutions of (1.3). They obey a set of $n + 1$ polynomial equations of the fourth order. We analyze the stability of the fixed point solutions by imposing the following restriction

$$(R) \quad \det \left(\frac{\partial f_i}{\partial \dot{h}^j}(\mathbf{0}, v) \right) \neq 0, \quad (1.8)$$

which guarantees the local resolution of eqs. (1.7) in the vicinity of the point $(\mathbf{0}, v) \in \mathbb{R}^{2n}$: $\dot{h}^i = \varphi^i(h)$ with $\varphi^i(v) = 0$, $i = 1, \dots, n$. Here $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$.

We also impose another restriction on v :

$$\sum_{k=1}^n v^k \neq 0, \quad (1.9)$$

which means that the solutions with constant volume scale factor are not considered here. We note that a solution with $\sum_{k=1}^n v^k = 0$ obeying (1.4) gives an enormously big value of the variation of G : $\dot{G}/G = 3H$, where H is the Hubble parameter, see Remark 5 in Section 4 below. This value of \dot{G} contradicts to the observational restrictions, e.g. (1.5). We remind that the present value of H is $(6.929 \pm 0.157) \cdot 10^{-11} \text{ year}^{-1}$ [31] (with 95% confidence level).

The main result of the paper is the following one: fixed point solutions $h(t) = v$ to eqs. (1.6) and (1.7), which obey restrictions (1.8) and (1.9), are stable if and only if $\sum_{k=1}^n v^k > 0$. This result is in agreement with the approach of S. Pavluchenko from ref. [22], see Remark 2 in Section 3 below.

The paper is organized as follows. In Section 2 the equations of motion for D -dimensional EGB model are considered. For diagonal cosmological metrics the equations of motion are equivalent to a set of Lagrange equations corresponding to a certain “effective” Lagrangian. The Lagrange equations for a certain choice of the lapse function (corresponding to the synchronous time variable) are reduced to the set of eqs. (1.6), (1.7). Section 3 is devoted to analysis of stability of the exponential solutions with constant Hubble-like parameters: here a set of equations for perturbations $\delta h^i(t)$ (obtained in linear approximation) is studied and general solution to these equations is found. The main proposition on stability of exponential solutions (Proposition 2) is proved. In Section 4 some examples of stable cosmological solutions with exponential behavior of scale factors are presented.

2 The model

2.1 The set-up

Here we consider the manifold

$$M = (t_-, t_+) \times M_1 \times \dots \times M_n, \quad (2.1)$$

with the metric

$$g = -e^{2\gamma(t)} dt \otimes dt + \sum_{i=1}^n e^{2\beta^i(t)} dy^i \otimes dy^i, \quad (2.2)$$

where $i = 1, \dots, n$; M_1, \dots, M_n are one-dimensional manifolds (either \mathbb{R} or S^1) and $n > 3$. The functions $\gamma(t)$ and $\beta^i(t)$, $i = 1, \dots, n$, are smooth on (t_-, t_+) .

For physical applications we put $M_1 = M_2 = M_3 = \mathbb{R}$, while M_4, \dots, M_n may be considered to be compact ones (i.e. coinciding with S^1).

The integrand in (1.1), when the metric (2.2) is substituted, reads as follows

$$\sqrt{|g|} \{ \alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g] \} = L + \frac{df}{dt}, \quad (2.3)$$

where

$$L = \alpha_1 (e^{-\gamma+\gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j - 2\Lambda e^{\gamma+\gamma_0}) - \frac{1}{3} \alpha_2 e^{-3\gamma+\gamma_0} G_{ijkl} \dot{\beta}^i \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l, \quad (2.4)$$

$\gamma_0 = \sum_{i=1}^n \beta^i$ and

$$G_{ij} = \delta_{ij} - 1, \quad (2.5)$$

$$G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl} \quad (2.6)$$

are respectively the components of two metrics on \mathbb{R}^n [15, 16]. The first one is “minisupermetric” - 2-metric of pseudo-Euclidean signature and the second one is the Finslerian 4-metric [15, 16]. Here we denote $\dot{A} = dA/dt$ etc. The function $f(t)$ in (2.3) is irrelevant for our consideration (see [15, 16]).

In derivation of (2.4) the following identities [15, 16] were used:

$$G_{ij} v^i v^j = \sum_{i=1}^n (v^i)^2 - \left(\sum_{i=1}^n v^i \right)^2 = S_2 - S_1^2, \quad (2.7)$$

$$G_{ijkl} v^i v^j v^k v^l = S_1^4 - 6S_1^2 S_2 + 3S_2^2 + 8S_1 S_3 - 6S_4. \quad (2.8)$$

Here and in what follows $S_k = S_k(v) = \sum_{i=1}^n (v^i)^k$.

The definitions (2.5) and (2.6) imply

$$G_{ij} v^i v^j = -2 \sum_{i < j} v^i v^j, \quad (2.9)$$

$$G_{ijkl} v^i v^j v^k v^l = 24 \sum_{i < j < k < l} v^i v^j v^k v^l. \quad (2.10)$$

The equations of motion corresponding to the action (1.1) have the following form

$$\mathcal{E}_{MN} = \alpha_1 \mathcal{E}_{MN}^{(1)} + \alpha_2 \mathcal{E}_{MN}^{(2)} = 0, \quad (2.11)$$

where

$$\mathcal{E}_{MN}^{(1)} = R_{MN} - \frac{1}{2}Rg_{MN} + \Lambda g_{MN}, \quad (2.12)$$

$$\begin{aligned} \mathcal{E}_{MN}^{(2)} = & 2(R_{MPQS}R_N^{PQS} - 2R_{MP}R_N^P \\ & - 2R_{MPNQ}R^{PQ} + RR_{MN}) - \frac{1}{2}\mathcal{L}_2g_{MN}. \end{aligned} \quad (2.13)$$

It may be shown (along a line as it was done in [16] for the case $\Lambda = 0$) that the field eqs. (2.11) for the metric (2.2) are equivalent to the Lagrange equations corresponding to the Lagrangian L from (2.4).

Thus, eqs. (2.11) read as follows

$$\alpha_1(G_{ij}\dot{\beta}^i\dot{\beta}^j + 2\Lambda e^{2\gamma}) - \alpha_2 e^{-2\gamma}G_{ijkl}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l = 0, \quad (2.14)$$

$$\frac{d}{dt}[2\alpha_1 G_{ij}e^{-\gamma+\gamma_0}\dot{\beta}^j - \frac{4}{3}\alpha_2 e^{-3\gamma+\gamma_0}G_{ijkl}\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l] - L = 0, \quad (2.15)$$

$i = 1, \dots, n$; and L is defined in (2.4).

Now we put $\gamma = 0$. By introducing ‘‘Hubble-like’’ variables $h^i = \dot{\beta}^i$, eqs. (2.14) and (2.15) may be rewritten as follows

$$E = E(h) \equiv G_{ij}h^ih^j + 2\Lambda - \alpha G_{ijkl}h^ih^jh^kh^l = 0, \quad (2.16)$$

$$U_i = U_i(\dot{h}, h) \equiv \frac{dL_i}{dt} + \left(\sum_{j=1}^n h^j\right)L_i - L_0 = 0, \quad (2.17)$$

where $\alpha = \alpha_1/\alpha_2$,

$$L_0 = G_{ij}h^ih^j - 2\Lambda - \frac{1}{3}\alpha G_{ijkl}h^ih^jh^kh^l, \quad (2.18)$$

and

$$L_i = L_i(h) = 2G_{ij}h^j - \frac{4}{3}\alpha G_{ijkl}h^jh^kh^l, \quad (2.19)$$

$i = 1, \dots, n$. Thus, we are led to the autonomous system of the first-order differential equations on $h^1(t), \dots, h^n(t)$ (see [15, 16] for $\Lambda = 0$).

Due to (2.16) we have

$$L_0 = \frac{2}{3}(G_{ij}h^ih^j - 4\Lambda). \quad (2.20)$$

In what follows we will use instead of (2.16), (2.17) an equivalent set of equations: (2.16) and

$$Y_i = Y_i(\dot{h}, h) \equiv \frac{dL_i}{dt} + \left(\sum_{j=1}^n h^j\right)L_i - \frac{2}{3}(G_{ij}h^ih^j - 4\Lambda) = 0. \quad (2.21)$$

We note that the following identity is valid

$$U_i(\dot{h}, h) = Y_i(\dot{h}, h) - \frac{1}{3}E(h), \quad (2.22)$$

$i = 1, \dots, n$.

Eqs. (2.16) and (2.21) are dependent, since

$$h^i Y_i = \frac{dE}{dt} + \frac{4}{3} \left(\sum_{j=1}^n h^j \right) E. \quad (2.23)$$

This identity may be proved by using two relations:

$$h^i \frac{dL_i}{dt} = \frac{dE}{dt}, \quad (2.24)$$

$$h^i L_i = L_0 + \frac{4}{3} E, \quad (2.25)$$

following from (2.16) and (2.19).

2.2 Useful relations

In what follows we use the definitions

$$B = B(v) = G_{ijks} v^i v^j v^k v^s, \quad A_i = A_i(v) = G_{ijkl} v^j v^k v^l. \quad (2.26)$$

For isotropic case

$$v = (v^i) = (H, \dots, H) \quad (2.27)$$

we get

$$B = n(n-1)(n-2)(n-3)H^4, \quad A_i = (n-1)(n-2)(n-3)H^3, \quad (2.28)$$

$i = 1, \dots, n$.

Here we deal with the ansatz which contain two Hubble parameters

$$v = (v^i) = (H, \dots, H, h, \dots, h) \quad (2.29)$$

where H appears m -times and h appears l -times, $n = m + l$. In what follows we adopt the following agreement for indices: $\mu, \nu, \dots = 1, \dots, m$; $\alpha, \beta, \dots = m + 1, \dots, n$. Thus, $v^\mu = H$ and $v^\alpha = h$.

We obtain

$$B = m(m-1)(m-2)(m-3)H^4 + 4m(m-1)(m-2)lH^3h + 6m(m-1)l(l-1)H^2h^2 + 4ml(l-1)(l-2)Hh^3 + l(l-1)(l-2)(l-3)h^4 \quad (2.30)$$

and

$$A_H \equiv A_\mu = (m-1)(m-2)(m-3)H^3 + 3(m-1)(m-2)lH^2h + 3(m-1)l(l-1)Hh^2 + l(l-1)(l-2)h^3, \quad (2.31)$$

$$A_h \equiv A_\alpha = m(m-1)(m-2)H^3 + 3m(m-1)(l-1)H^2h + 3m(l-1)(l-2)Hh^2 + (l-1)(l-2)(l-3)h^3. \quad (2.32)$$

We also denote

$$S_{ij} = G_{ijks} v^k v^s, \quad (2.33)$$

We note that $S_{ij} = S_{ji}$ and $S_{ii} = 0$. For isotropic case (2.27) we obtain

$$S_{ij} = (n-2)(n-3)H^2, \quad i \neq j. \quad (2.34)$$

For the the ansatz (2.29) we obtain

$$S_{HH} = (m-2)(m-3)H^2 + 2(m-2)lHh + l(l-1)h^2, \quad (2.35)$$

$$S_{Hh} = (m-1)(m-2)H^2 + 2(m-1)(l-1)Hh + (l-1)(l-2)h^2, \quad (2.36)$$

$$S_{hh} = m(m-1)H^2 + 2m(l-2)Hh + (l-2)(l-3)h^2. \quad (2.37)$$

Here we denote: $S_{\mu\nu} = S_{HH}$ for $\mu \neq \nu$; $S_{\mu\alpha} = S_{\alpha\mu} = S_{Hh}$; $S_{\alpha\beta} = S_{hh}$ for $\alpha \neq \beta$.

2.3 Polynomial equations for solutions with constant h^i

Let us consider the following solutions to eqs. (2.16) and (2.21)

$$h^i(t) = v^i, \quad (2.38)$$

with constant v^i , which correspond to the solutions

$$\beta^i = v^i t + \beta_0^i, \quad (2.39)$$

where β_0^i are constants, $i = 1, \dots, n$.

In this case we obtain the metric (2.2) with the exponential dependence of scale factors

$$g = -dt \otimes dt + \sum_{i=1}^n B_i^2 e^{2v^i t} dy^i \otimes dy^i, \quad (2.40)$$

where $B_i > 0$ are arbitrary constants.

For the fixed point $v = (v^i)$ we have the set of polynomial equations

$$E = E(v) = G_{ij} v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l = 0, \quad (2.41)$$

$$Y_i = Y_i(\mathbf{0}, v) = \left(\sum_{j=1}^n v^j \right) L_i(v) - \frac{2}{3} G_{kj} v^k v^j + \frac{8}{3} \Lambda = 0, \quad (2.42)$$

where L_i is defined in (2.19), $i = 1, \dots, n$. For $n > 3$ this is the set of forth-order polynomial equations.

Here and in what follows we use relations (2.7), (2.8) and the following formulas

$$v_i = G_{ij} v^j = v^i - S_1, \quad (2.43)$$

$$\begin{aligned} A_i &= G_{ijkl} v^j v^k v^l = S_1^3 + 2S_3 - 3S_1 S_2 \\ &\quad + 3(S_2 - S_1^2) v^i + 6S_1 (v^i)^2 - 6(v^i)^3, \end{aligned} \quad (2.44)$$

$i = 1, \dots, n$ ($S_k = \sum_{i=1}^n (v^i)^k$).

Proposition 1. *For any solution $v = (v^1, \dots, v^n)$ to polynomial eqs. (2.41), (2.42) with $n > 3$ there are no more than three different numbers among v^1, \dots, v^n , if $\sum_{i=1}^n v^i \neq 0$.*

Proof. Let us suppose that there exists a non-trivial solution $v = (v^1, \dots, v^n)$ with more than three different numbers among v^1, \dots, v^n . Due to (2.44), (2.42) and $\sum_{i=1}^n v^i \neq 0$ we get $C_0 + C_1 v^i + C_2 (v^i)^2 + C_3 (v^i)^3 = 0$, $i = 1, \dots, n$, with some real numbers C_0, C_1, C_2 and $C_3 \neq 0$. Let us consider the cubic equation $C_0 + C_1 x + C_2 x^2 + C_3 x^3 = 0$. Any number v^i obeys this equation and hence at most three numbers among v^i may be different. Thus, we are led to a contradiction. The proposition is proved. The case $\Lambda = 0$ was considered earlier in [15, 16].

Remark 1. *In pure Einstein case ($\alpha = 0$) with $\Lambda > 0$ we get two exponential solutions with $v^1 = \dots = v^n = H$ and $n(n-1)H^2 = 2\Lambda > 0$; solution with $H > 0$ is an attractor for cosmological solutions with diagonal metrics, as $t \rightarrow +\infty$, see [32] and [33] (for $\varphi = 0$). Thus in this case ($\alpha = 0$) we have an isotropisation for $t \rightarrow +\infty$, while for $t \rightarrow +0$ we have Kasner-like behaviour of scale factors near the singularity: $a_i(t) \sim t^{p_i}$ with Kasner parameters p_1, \dots, p_n obeying: $\sum_{i=1}^n p_i = \sum_{i=1}^n p_i^2 = 1$. In the case of EGB model with Λ -term we have for certain Λ and α isotropic exponential solutions with $v^1 = \dots = v^n = H$ (see Section 4 below), but we also may have partially anisotropic (PA) solutions, which obey $\sum_{i=1}^n v^i \neq 0$, with: $v = (H, \dots, H, h, \dots, h)$ or $v = (H, \dots, H, h, \dots, h, z, \dots, z)$, and also solutions with $\sum_{i=1}^n v^i = 0$ may take place. For $\sum_{i=1}^n v^i = 0$ (and certain Λ and α) one may obtain examples of totally anisotropic exponential solutions with non-coinciding parameters among v^1, \dots, v^n . Some of the exponential PA solutions are stable (see below) and they are attractors of certain subclasses of general solutions. The appearance of three (or less) independent scale factors in the model under consideration is a feature of exponential (e.g. attractor) solutions, when restriction $\sum_{i=1}^n v^i \neq 0$ is imposed. We also note that the metric (2.40) may be analyzed on symmetries (apparent or hidden) by using the results of ref. [34], i.e. Killing vectors, isometry group, coset structure G/H etc, may be presented. The Proposition 2 may be also generalized to the Lovelock case [35] – that may be a subject of a separate publication.*

Now let us consider the ansatz (2.29) with two Hubble parameters H and h with two restrictions imposed

$$mH + lh \neq 0, \quad H \neq h. \quad (2.45)$$

In this case the set of $n+1$ eqs. (2.16), (2.17) is equivalent to the set of three equations

$$E = 0, \quad Y_H = 0, \quad Y_h = 0, \quad (2.46)$$

where $Y_H = Y_\mu$, $Y_h = Y_\alpha$ ($\mu = 1, \dots, m$, $\alpha = m+1, \dots, n$).

Due to (2.44) we have

$$A_H - A_h = (H - h)[3(S_2 - S_1^2) + 6S_1(H + h) - 6(H^2 + Hh + h^2)], \quad (2.47)$$

and hence, by using (2.19), (2.43), we obtain

$$Y_H - Y_h = (H - h)(mH + lh)[2 + 4\alpha Q(H, h)], \quad (2.48)$$

where

$$Q(H, h) = (m-1)(m-2)H^2 + 2(m-1)(l-1)Hh + (l-1)(l-2)h^2. \quad (2.49)$$

For $m > 1$ and $l > 1$ the quadratic form has the signature $(-, +)$. Due to $mH + lh \neq 0$ the set of eqs. (2.46) is equivalent to another set of equations

$$E = 0, \quad Y_H - Y_h = 0, \quad mHY_H + lhY_h = 0, \quad (2.50)$$

According to (2.23) $E = 0$ implies $h^i Y_i = mHY_H + lhY_h = 0$ and hence the third equation in (2.50) may be omitted. Using restrictions (2.45), relations (2.30) and (2.48) we reduce the set eqs. (2.50) to the following set of equations

$$\begin{aligned} E = mH^2 + lh^2 - (mH + lh)^2 + 2\Lambda - \alpha[m(m-1)(m-2)(m-3)H^4 \\ + 4m(m-1)(m-2)lH^3h + 6m(m-1)l(l-1)H^2h^2 \\ + 4ml(l-1)(l-2)Hh^3 + l(l-1)(l-2)(l-3)h^4] = 0, \end{aligned} \quad (2.51)$$

$$1 + 2\alpha Q(H, h) = 0, \quad (2.52)$$

where $Q(H, h)$ is defined in (2.49).³ Thus, for the anisotropic solutions with two different Hubble parameters H and h and non-static volume factor (see (2.29) and (2.45)) the set $(n+1)$ polynomial eqs. of fourth order (2.41) and (2.42) is equivalent to the set of two eqs. (2.51) and (2.52) of fourth and second order respectively.

3 Stability of fixed point solutions $h^i(t) = v^i$

Here we study the stability of static solutions $h^i(t) = v^i$ to eqs. (2.16) and (2.17) in linear approximation in perturbations. We put

$$h^i(t) = v^i + \delta h^i(t), \quad (3.1)$$

$i = 1, \dots, n$. By substitution (3.1) into eqs. (2.16) and (2.17) we obtain in linear approximation the following relations for perturbations δh^i

$$C_i(v)\delta h^i = 0, \quad (3.2)$$

$$L_{ij}(v)\delta \dot{h}^j = B_{ij}(v)\delta h^j, \quad (3.3)$$

where

$$C_i = C_i(v) = 2v_i - 4\alpha G_{ijk}s v^j v^k v^s, \quad (3.4)$$

$$L_{ij} = L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijk}s v^k v^s, \quad (3.5)$$

$$B_{ij} = B_{ij}(v) = -\left(\sum_{k=1}^n v^k\right)L_{ij}(v) - L_i(v) + \frac{4}{3}v_j. \quad (3.6)$$

³For general scheme of reduction see [20].

We remind that $v_i = G_{ij}v^j$, $L_i(v) = 2v_i - \frac{4}{3}\alpha G_{ijk}s v^j v^k v^s$ and $i, j, k, s = 1, \dots, n$.

We put the following restriction on the matrix $L = (L_{ij}(v))$

$$(R) \quad \det(L_{ij}(v)) \neq 0, \quad (3.7)$$

i.e. the matrix L should be invertible.

Here we restrict ourselves by exponential solutions (2.40) with non-static volume factor, which is proportional to $\exp(\sum_{i=1}^n v^i t)$, i.e. we put

$$K = K(v) = \sum_{i=1}^n v^i \neq 0. \quad (3.8)$$

Then we get from eq. (2.42)

$$L_i(v) = L_1 = \frac{2}{3} \left(\sum_{k=1}^n v^k \right)^{-1} (G_{ij}v^i v^j - 4\Lambda). \quad (3.9)$$

Due to definition (2.19) we have

$$\alpha A_i = \alpha G_{ijk}s v^j v^k v^s = \frac{3}{4} (2v_i - L_1) \quad (3.10)$$

and hence

$$C_i(v) = 2v_i - 4\alpha A_i = -4v_i + 3L_1. \quad (3.11)$$

We rewrite relation (3.6) as

$$B_{ij} = -\left(\sum_{k=1}^n v^k\right) L_{ij}(v) + \hat{B}_{ij}, \quad \hat{B}_{ij} = -L_i(v) + \frac{4}{3}v_j. \quad (3.12)$$

Due to $L_i(v) = L_1$ and (3.2) we get

$$\hat{B}_{ij}\delta h^j = -L_1 \sum_{j=1}^n \delta h^j + \frac{4}{3}v_j \delta h^j = -\frac{1}{3}C_j(v)\delta h^j = 0. \quad (3.13)$$

Hence eq. (3.3) reads

$$L_{ij}(v)\delta \dot{h}^j = -\left(\sum_{k=1}^n v^k\right) L_{ij}\delta h^j, \quad (3.14)$$

or, equivalently,

$$\delta \dot{h}^i = -\left(\sum_{k=1}^n v^k\right) \delta h^i, \quad (3.15)$$

$i = 1, \dots, n$. Here we used the restriction (3.7).

Thus, the set of linear equations on perturbations (3.2), (3.3) is equivalent to the set of linear eqs. (3.2), (3.15), which has the following solution

$$\delta h^i = A^i \exp(-K(v)t), \quad (3.16)$$

$$\sum_{i=1}^n C_i(v) A^i = 0. \quad (3.17)$$

$i = 1, \dots, n$. We remind that $K(v) = \sum_{k=1}^n v^k$.

Due to (3.16) that the following proposition is valid.

Proposition 2. *The fixed point solution $(h^i(t)) = (v^i)$ ($i = 1, \dots, n$; $n > 3$) to eqs. (2.16), (2.17) obeying restrictions (3.7), (3.8) is stable under perturbations (3.1) (as $t \rightarrow +\infty$) if $K(v) = \sum_{k=1}^n v^k > 0$ and it is unstable (as $t \rightarrow +\infty$) if $K(v) = \sum_{k=1}^n v^k < 0$.*

It follows from (2.34) that in the isotropic case the matrix (3.5) reads

$$L_{ij} = \varphi(H) G_{ij}, \quad \varphi(H) = 2 + 4\alpha(n-2)(n-3)H^2. \quad (3.18)$$

Since the matrix $(G_{ij}) = (\delta_{ij} - 1)$ is invertible (or, non-degenerate one) for $n > 1$ (its inverse is $(G^{ij}) = (\delta^{ij} - \frac{1}{n-1})$) then the matrix (L_{ij}) is invertible if and only if $\varphi(H) \neq 0$.

Now let us consider the matrix (3.5) for the anisotropic case (2.29) with two Hubble parameters obeying (2.45).

For the the ansatz (2.29) we obtain

$$L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \quad (3.19)$$

$$L_{\mu\alpha} = L_{\alpha\mu} = -2 - 4\alpha S_{Hh}, \quad (3.20)$$

$$L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}). \quad (3.21)$$

Here S_{HH} , S_{Hh} and S_{hh} are defined in (2.35), (2.36) and (2.37), respectively. But here we have a remarkable coincidence (see (2.49))

$$Q(H, h) = S_{Hh}, \quad (3.22)$$

which implies $L_{\mu\alpha} = L_{\alpha\mu} = 0$ due to eq. (2.52). Thus under restrictions (2.45) assumed the matrix (L_{ij}) has a block-diagonal form

$$(L_{ij}) = \text{diag}(L_{\mu\nu}, L_{\alpha\beta}). \quad (3.23)$$

This matrix is invertible if and only if $m > 1$, $l > 1$ and

$$S_{HH} \neq -\frac{1}{2\alpha}, \quad S_{hh} \neq -\frac{1}{2\alpha}. \quad (3.24)$$

We remind that $m \times m$ matrix $(G_{\mu\nu})$ and $l \times l$ matrix $(G_{\alpha\beta})$ are invertible only for $m > 1$ and $l > 1$, respectively.

Remark 2. *Recently, in ref. [22] a criterion for stability of fixed point solutions in the model under consideration (and its extension to the Lovelock case) was used. In our notations (see Introduction) it reads:*

$$\frac{\partial \dot{h}^i}{\partial h^i}(v) = \frac{\partial \varphi^i}{\partial h^i}(v) < 0, \quad (3.25)$$

$i = 1, \dots, n$. It can be readily verified that for generic functions f_0, f_i in eqs. (1.6), (1.7) the criterion (3.25) is not a necessary and/or a sufficient condition for the stability of the fixed point solutions. Fortunately, for a special choice of functions, e.g. for $f_0(h) = E(h)$, $f_i(\dot{h}, h) = Y_i(\dot{h}, h) + \frac{1}{3}E(h) = U_i(\dot{h}, h)$ (see (2.22) and (3.13)), it gives a correct result since in this case

$$\frac{\partial \dot{h}^i}{\partial h^i}(v) = - \sum_{k=1}^n v^k, \quad (3.26)$$

$i = 1, \dots, n$. Relation (3.26) is also valid for $f_i(\dot{h}, h) = \lambda U_i(\dot{h}, h)$ with $\lambda \neq 0$, e.g. for the choice $\lambda = -1$ used in [22]. We also note that in our notations $2\Lambda = \Lambda_P$, where Λ_P is the Λ -term from ref. [22].

4 Examples

Here we consider several examples of exponential solutions and analyse their stability.

4.1 Isotropic solution

Let us consider the isotropic solution $v = (v^i) = (H, \dots, H)$ to eqs. (2.41), (2.42) for $n > 3$. Due to $G_{ij}v^i v^j = n(1-n)H^2$ and (2.28), eq. (2.41) reads as follows

$$2F(H^2) = n(n-1)H^2 + \alpha n(n-1)(n-2)(n-3)H^4 = 2\Lambda. \quad (4.1)$$

Eq. (2.42) is also equivalent to (4.1) due to relation

$$L_i = -2(n-1)H + \frac{4}{3}\alpha(n-1)(n-2)(n-3)H^3, \quad (4.2)$$

$i = 1, \dots, n$, which follows from (2.19), (2.28) and (2.43).

Let $\Lambda = 0$. The trivial solution $H = 0$ is valid for any α . This is the unique solution for $\alpha > 0$. For $\alpha < 0$ we have two non-trivial solutions [15, 16] with

$$H^2 = \frac{1}{|\alpha|(n-2)(n-3)}. \quad (4.3)$$

This solution was generalized in [19] to the case $\Lambda \neq 0$.

Let us consider the case of generic Λ in detail. First, we put $\alpha > 0$. Then, a solution to eq. (4.1) does exist if and only if $\Lambda \geq 0$. For $\Lambda = 0$ we get $H = 0$, while for $\Lambda > 0$ we have two non-zero solutions for H with $H^2 > 0$:

$$H^2 = \frac{-n(n-1) + \sqrt{\Delta}}{2\alpha n(n-1)(n-2)(n-3)}, \quad (4.4)$$

where

$$\Delta = n^2(n-1)^2 + 8\Lambda\alpha n(n-1)(n-2)(n-3). \quad (4.5)$$

Now we put $\alpha < 0$. A solution to eq. (4.1) exists only if $\Lambda \leq \Lambda_{cr}$, where

$$\Lambda_{cr} = -\frac{n(n-1)}{8\alpha(n-2)(n-3)} \quad (4.6)$$

is the maximum value of the function $F(H^2)$ from (4.1). For $0 < \Lambda < \Lambda_{cr}$ (and $\alpha < 0$) we have two solutions for H^2 (or four solutions for H) which are given by relation

$$H^2 = \frac{-n(n-1) \pm \sqrt{\Delta}}{2\alpha n(n-1)(n-2)(n-3)}. \quad (4.7)$$

For $\Lambda = \Lambda_{cr}$ and $\alpha < 0$ we get one solution for H^2 (or two solutions for H):

$$H^2 = H_{cr}^2 = -\frac{1}{2\alpha(n-2)(n-3)}. \quad (4.8)$$

The case $\Lambda = 0$ (and $\alpha < 0$) was mentioned above (two solutions for H^2 , or three - for H). For $\Lambda < 0$ (and $\alpha < 0$) we obtain one solution for H^2 (or two solutions for H):

$$H^2 = \frac{-n(n-1) - \sqrt{\Delta}}{2\alpha n(n-1)(n-2)(n-3)}. \quad (4.9)$$

Due to (3.18) the matrix (L_{ij}) is invertible for all solutions but $H = H_{cr}$ from (4.8) for $\alpha < 0$, since only in this case $\varphi(H) = 0$. The relation $H = H_{cr}$ takes place only for $\Lambda = \Lambda_{cr}$ and $\alpha < 0$ and hence this case will be excluded from our analysis. Since $K(v) = nH$, the trivial solution $H = 0$ for $\Lambda = 0$ should be also excluded from our consideration. It follows from Proposition 2 that all isotropic solutions $v = (v^i) = (H, \dots, H)$ obeying $H > 0$ and $H \neq H_{cr}$ for $\alpha < 0$ are stable while all isotropic solutions obeying $H < 0$ and $H \neq H_{cr}$ for $\alpha < 0$ are unstable.

Using (2.28), (2.43) and (3.4) we obtain $C_i(v) = -(n-1)H\varphi(H) \neq 0$, $i = 1, \dots, n$, for $H \neq 0$ and $H \neq H_{cr}$ for $\alpha < 0$. Under these restrictions on H , the solution for perturbations (3.16), (3.17) reads as follows

$$\delta h^i = A^i \exp(-nHt), \quad (4.10)$$

$$\sum_{i=1}^n A^i = 0, \quad (4.11)$$

$i = 1, \dots, n$. Relation (4.10) was obtained earlier in [22].

4.2 Anisotropic solutions with two Hubble parameters

In this subsection we consider several examples of anisotropic solutions to eqs. (2.41), (2.42) of the form $v = (H, \dots, H, h, \dots, h)$, where H the Hubble-like parameter corresponding to m -dimensional isotropic subspace with $m \geq 3$ and h is the Hubble-like parameter corresponding to l -dimensional isotropic subspace with $l > 1$. Here we put $H > 0$.

4.2.1 Solution for $m = 3$, $l = 2$ and $\Lambda = 0$.

Let us consider the case $m = 3$, $l = 2$, $\Lambda = 0$. We have the following solution to the set of polynomial eqs. (2.51), (2.52) with $H > 0$:

$$H = \frac{1}{6}(7 + 4 \cdot 10^{1/3} + 10^{2/3})^{1/2} \alpha^{-1/2} \approx 0.750173 \alpha^{-1/2}, \quad (4.12)$$

$$h = -\frac{1}{6}(7 - 0.5 \cdot 10^{1/3} + 10^{2/3})^{1/2} \alpha^{-1/2} \approx -0.541715 \alpha^{-1/2}. \quad (4.13)$$

It the approximate form this solution was found earlier in [17], in analytic form (different from (4.12), (4.13)) it was obtained in [19].

Using (2.35) and (2.37) we get

$$S_{HH} = 2h(2H + h) \approx -1.038610 \alpha^{-1}, \quad S_{hh} = 6H^2 \approx 3.376557 \alpha^{-1}. \quad (4.14)$$

Relations (3.24) are valid and hence the first restriction (3.7) is satisfied. The second restriction (3.8) is also satisfied since $K(v) = 3H + 2h > 0$. Thus, due to Proposition 2, the solution is stable in agreement with [22].

4.2.2 Solution for $m = l = 3$ and $\Lambda = 0$

Now we consider solutions with $m = 3$, $l = 3$ and $\Lambda = 0$. There are two solutions to eqs. (2.51), (2.52) with $H > 0$:

$$H_1 = \frac{1}{4}(\sqrt{5} - 1) \alpha^{-1/2}, \quad h_1 = \frac{1}{4}(-\sqrt{5} - 1) \alpha^{-1/2}, \quad (4.15)$$

and

$$H_2 = \frac{1}{4}(\sqrt{5} + 1) \alpha^{-1/2}, \quad h_2 = \frac{1}{4}(-\sqrt{5} + 1) \alpha^{-1/2}. \quad (4.16)$$

For the first solution we get

$$S_{HH} = \frac{3}{4}(\sqrt{5} + 1) \alpha^{-1}, \quad S_{hh} = \frac{3}{4}(-\sqrt{5} + 1) \alpha^{-1}, \quad (4.17)$$

while for the second one we obtain

$$S_{HH} = \frac{3}{4}(-\sqrt{5} + 1) \alpha^{-1}, \quad S_{hh} = \frac{3}{4}(\sqrt{5} + 1) \alpha^{-1}. \quad (4.18)$$

In both cases relations (3.24) are satisfied and hence the first restriction (3.7) is valid. The second restriction (3.8) is also valid for any of these solutions since $K(v_1) = 3H_1 + 3h_1 = -\frac{3}{2} \alpha^{-1/2} < 0$ and $K(v_2) = 3H_2 + 3h_2 = \frac{3}{2} \alpha^{-1/2} > 0$. According to Proposition 2 the first solution (4.15) is unstable, while the second one (4.16) is stable.

4.2.3 Solution for $m = 11$, $l = 16$ and $\Lambda = 0$

For $\Lambda = 0$ the solution (2.40) with $v = (v^i)$ from (2.29), $m = 11$, $l = 16$ and

$$H = \frac{1}{\sqrt{15\alpha}}, \quad h = -\frac{1}{2\sqrt{15\alpha}} \quad (4.19)$$

was found in [21]. This solution describes the zero variation of the effective cosmological constant G .

The calculations give us

$$S_{HH} = -\frac{4}{5}\alpha^{-1}, \quad S_{hh} = \frac{1}{10}\alpha^{-1}. \quad (4.20)$$

Due to (3.24) the symmetric matrix (L_{ij}) , which has a block-diagonal form, is invertible, i.e. the condition (3.7) is satisfied.

Using (3.9) and (3.11) we find $(C_i) = (C_\mu = 12H, C_\alpha = 18H)$. From (3.16) we get the following solution for perturbations

$$\delta h^i = A^i \exp(-3Ht), \quad (4.21)$$

$$2 \sum_{\mu=1}^{11} A^\mu + 3 \sum_{\alpha=12}^{27} A^\alpha = 0, \quad (4.22)$$

where $H = \frac{1}{\sqrt{15\alpha}}$, $i = 1, \dots, 27$. Thus, the solution (4.19) is stable, as $t \rightarrow +\infty$.

4.2.4 Solution for $m = 15$, $l = 6$ and $\Lambda = 0$

Now we consider another exponential solution (2.40) from [21] with $v = (v^i)$ from (2.29), $m = 15$, $l = 6$, $\Lambda = 0$ and

$$H = \frac{1}{6}\alpha^{-1/2}, \quad h = -\frac{1}{3}\alpha^{-1/2}. \quad (4.23)$$

We get

$$S_{HH} = -\alpha^{-1}, \quad S_{hh} = \frac{1}{2}\alpha^{-1}. \quad (4.24)$$

According to (3.24) the symmetric block-diagonal matrix (L_{ij}) is non-degenerate one.

By using (3.9) and (3.11) we get $(C_i) = (C_\mu = \frac{14}{3}, C_\alpha = \frac{20}{3})$. Due to (3.16) the solution for perturbations reads

$$\delta h^i = A^i \exp(-3Ht) = A^i \exp(-\frac{1}{2}\alpha^{-1/2}t), \quad (4.25)$$

$$7 \sum_{\mu=1}^{15} A^\mu + 10 \sum_{\alpha=16}^{21} A^\alpha = 0, \quad (4.26)$$

$i = 1, \dots, 21$. Hence, the solution (4.23) is stable as $t \rightarrow +\infty$.

Remark 3. *The stability of this solution as well as the previous one was also proved in ref. [23] by using rather tedious calculations based on relations (3.3) and (3.6) without using the identity (3.13).*

4.2.5 Solutions with $m \geq 3$, $l > 1$ and certain $\Lambda > 0$

Here we consider the following solution to eqs. (2.51), (2.52) for $m > 2$, $l > 1$ and $\alpha < 0$:

$$H^2 = -\frac{1}{2\alpha(m-1)(m-2)}, \quad h = 0, \quad (4.27)$$

which is valid for

$$\Lambda = -\frac{m(m+1)}{8\alpha(m-1)(m-2)} > 0. \quad (4.28)$$

We get from (2.35) and (2.37)

$$S_{HH} = (m-2)(m-3)H^2 = -\frac{m-3}{2\alpha(m-1)} \neq -\frac{1}{2\alpha} \quad (4.29)$$

and

$$S_{hh} = m(m-1)H^2 = -\frac{m}{2\alpha(m-2)} \neq -\frac{1}{2\alpha}, \quad (4.30)$$

which implies the fulfilment of the restriction (3.7) (here $m > 2$ and $l > 1$). Since $K(v) = mH$ we get from Proposition 2 that the cosmological solution (2.40) with H , h from (4.27) is stable for $H > 0$ and unstable for $H < 0$.

4.3 A subclass of solutions with zero variation of G

The 4d effective gravitational constant is proportional to inverse volume scale factor of the internal space (see [27, 28, 29]), i.e.

$$G \sim \prod_{i=4}^n [a_i(t)]^{-1}, \quad (4.31)$$

where $a_i(t) = \exp(\beta^i(t))$.

Remark 4. Here $G = G_{eff}^J(t)$ is four-dimensional effective gravitational constant which appear in (multidimensional analogue of) the so-called Brans-Dicke-Jordan (or simply Jordan) frame [36]. In this case the physical 4-dimensional metric $g^{(4)}$ is defined as 4-dimensional section of the multidimensional metric g , i.e. $g^{(4)} = g^{(4,J)}$, where $g = g^{(4,J)} + \sum_{i=4}^n a_i^2(t) dy^i \otimes dy^i$. When the Einstein-Pauli (or simply Einstein) frame is used, we put $g^{(4)} = g^{(4,E)} = (\prod_{i=4}^n a_i(t)) g^{(4,J)}$ [36, 37] and hence we get the effective gravitational constant to be an exact constant: $G_{eff}^E = G_{eff}^J(t) \prod_{i=4}^n a_i(t) = \text{const}$ [36].

For the solutions (2.40) we obtain the following relations

$$G(t) = G(0) \exp(-K_{int}t), \quad K_{int}(v) = \sum_{i=4}^n v^i, \quad (4.32)$$

which imply

$$\frac{\dot{G}}{G} = -K_{int}(v). \quad (4.33)$$

Now, let us consider a subclass of cosmological solutions (2.40) which obey restriction (3.7) and describe an exponential isotropic expansion of 3-dimensional flat factor-space with $v^1 = v^2 = v^3 = H > 0$ with zero variation of G . Then we get from (4.33) $K_{int}(v) = 0$ and hence $K(v) = \sum_{i=1}^n v^i = 3H + K_{int}(v) = 3H > 0$. According to Proposition 2 any solution from this subclass is stable. Three solutions from the previous subsection: (4.19), (4.23) and (4.27) with $m = 3$ (and $l > 1$) belong to this subclass.

Remark 5. *It should be noted that for $K(v) = 0$ and $v^1 = v^2 = v^3 = H > 0$ we obtain $K_{int}(v) = -3H$ and hence $\frac{\dot{G}}{G} = 3H > 0$.*

5 Conclusions

We have considered the $(n + 1)$ -dimensional Einstein-Gauss-Bonnet (EGB) model with the Λ -term. By using the ansatz with diagonal cosmological metrics, we have studied the stability of solutions with exponential dependence of scale factors $a_i \sim \exp(v^i t)$, $i = 1, \dots, n$, with respect to synchronous time variable t in dimension $D > 4$.

The problem was reduced to the analysis of stability of the fixed point solutions $h^i(t) = v^i$ to eqs. (2.16) and (2.21), where $h^i(t)$ are Hubble-like parameters.

In this paper a set of equations for perturbations δh^i was considered (in linear approximation) and general solution to these equations was found. We have proved (in Proposition 2) that the solutions with non-static volume factor, i.e. with $K(v) = \sum_{k=1}^n v^k \neq 0$, which obey restriction (3.7), are stable if $K(v) > 0$ while they are unstable if $K(v) < 0$.

We have also proved (in Proposition 1) that for any exponential solution with $v = (v^1, \dots, v^n)$ there are no more than three different numbers among v^1, \dots, v^n , if $\sum_{i=1}^n v^i \neq 0$.

Here we have presented several examples of stable cosmological solutions with exponential behavior of scale factors. Among them the isotropic solution $v = (H, \dots, H)$ and several anisotropic solutions with two Hubble parameters $v = (H, \dots, H, h, \dots, h)$ were considered. The isotropic solution is stable if $H > 0$ and $H \neq H_{cr}$ for $\alpha < 0$ (see (4.8)). For the anisotropic case our examples deal with the Hubble-like parameter $H > 0$ corresponding to m -dimensional flat subspace with $m \geq 3$ and the Hubble-like parameter h corresponding to l -dimensional flat subspace with $l > 1$. This subclass of (anisotropic) solutions contains the following cases: i) $m = 3$, $l = 2$, $\Lambda = 0$; ii) $m = l = 3$, $\Lambda = 0$; iii) $m = 11$, $l = 16$, $\Lambda = 0$; iv) $m = 15$, $l = 6$, $\Lambda = 0$; v) $m \geq 3$, $l > 1$, $\Lambda > 0$. We have also shown that general solutions with $v^1 = v^2 = v^3 = H > 0$ and zero variation of the effective gravitational constant are stable if the restriction (3.7) is obeyed.

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